1 A Brief Definition of Set Theory

Set theory is the foundational branch of mathematical logic that studies sets, which are abstract, well-defined collections of distinct objects (called elements or members). In axiomatic set theory, sets themselves are the primary ontological objects, and all mathematical concepts (numbers, relations, functions) are constructed using sets. The fundamental relation in this theory is the membership relation, denoted by $x \in A$, meaning the object x is an element of the set A.

1.1 Fundamental Set Operations

In set theory, standard arithmetic operations are replaced by operations that combine or alter sets.

1. **Addition (Union):** $A \cup B = \{x : x \in A \lor x \in B\}$. The set of elements in A, or in B, or in both.
2. **Multiplication (Intersection):** $A \cap B = \{x : x \in A \land x \in B\}$. The set of elements common to both A and B (often viewed as boolean multiplication).
3. **Multiplication (Cartesian Product):** $A \times B = \{(a, b) : a \in A \land b \in B\}$. The set of all ordered pairs. (This is the combinatorial analogue to arithmetic multiplication).
4. **Subtraction (Set Difference):** $A \setminus B = \{x : x \in A \land x \notin B\}$. The set of elements in A that are not in B.
5. **Symmetric Difference:** $A \triangle B = \{x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A)\}$. The set of elements in exactly one of A or B, but not both.

1.2 Theoretical Proofs of Set Properties (Element)

To prove set equalities purely theoretically without numbers, we use the method of double inclusion: to show $X = Y$, we must prove $X \subseteq Y$ and $Y \subseteq X$. This is done by showing $x \in X \iff x \in Y$ using propositional logic.

1.3 Distributivity of Intersection over Union (Multiplication over Addition)

Theorem 1.1 Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Let x be an arbitrary element.

$$x \in A \cap (B \cup C) \iff x \in A \land x \in (B \cup C) \quad \text{(Definition of Intersection)}$$

$$
\begin{aligned}
&\iff x \in A \land (x \in B \lor x \in C) && \text{(Definition of Union)}\\
&\iff (x \in A \land x \in B) \lor (x \in A \land x \in C) && \text{(Logical Distributivity)}\\
&\iff (x \in A \cap B) \lor (x \in A \cap C) && \text{(Definition of Intersection)}\\
&\iff x \in (A \cap B) \cup (A \cap C) && \text{(Definition of Union)}
\end{aligned}
$$

Since $x \in A \cap (B \cup C) \iff x \in (A \cap B) \cup (A \cap C)$, the sets are equal. ∎

1.4 De Morgan's Law for Subtraction

Theorem 1.2 Show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof. Let x be an arbitrary element.

$$
\begin{aligned}
x \in A \setminus (B \cup C) &\iff x \in A \land x \notin (B \cup C) && \text{(Definition of Difference)}\\
&\iff x \in A \land \neg(x \in B \lor x \in C) && \text{(Definition of Union)}\\
&\iff x \in A \land (x \notin B \land x \notin C) && \text{(De Morgan's Law of Logic)}\\
&\iff (x \in A \land x \notin B) \land (x \in A \land x \notin C) && \text{(Idempotence and Associativity)}\\
&\iff (x \in A \setminus B) \land (x \in A \setminus C) && \text{(Definition of Difference)}\\
&\iff x \in (A \setminus B) \cap (A \setminus C) && \text{(Definition of Intersection)}
\end{aligned}
$$

Thus, the sets contain precisely the same elements. ∎

1.5 Equivalence of Symmetric Difference Definitions

Theorem 1.3 Show that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Note: By definition, $A \triangle B = (A \setminus B) \cup (B \setminus A)$. We prove this is equal to the union minus the intersection.

Proof. Let x be an arbitrary element. We will establish a chain of logical biconditionals

$$
\begin{aligned}
x \in (A \setminus B) \cup (B \setminus A) &\iff x \in (A \setminus B) \lor x \in (B \setminus A)\\
&\iff (x \in A \land x \notin B) \lor (x \in B \land x \notin A)\\
&\iff ((x \in A \land x \notin B) \lor x \in B) \land ((x \in A \land x \notin B) \lor x \notin A)\\
&\quad \text{(Distributivity of } \lor \text{ over } \land \text{)}\\
&\iff (x \in A \lor x \in B) \land (x \notin B \lor x \in B)\\
&\quad \land (x \in A \lor x \notin A) \land (x \notin B \lor x \notin A)\\
&\iff (x \in A \lor x \in B) \land \text{True} \land \text{True} \land \neg(x \in B \land x \in A)\\
&\quad \text{(Law of Excluded Middle and De Morgan's)}
\end{aligned}
$$

$$\begin{aligned}
&\iff (x \in A \lor x \in B) \land \neg(x \in A \land x \in B) \\
&\iff x \in (A \cup B) \land x \notin (A \cap B) \\
&\iff x \in (A \cup B) \setminus (A \cap B)
\end{aligned}$$

Therefore, the symmetric difference of two sets is the elements in their union excluding those in their intersection. ■

1.6 Distributivity of Cartesian Product (Multiplication) over Union (Addition)

Theorem 1.4 Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof. Let (x, y) be an arbitrary ordered pair.

$$\begin{aligned}
(x,y) \in A \times (B \cup C) &\iff x \in A \land y \in (B \cup C) \\
&\qquad \text{(Definition of Cartesian Product)} \\
&\iff x \in A \land (y \in B \lor y \in C) && \text{(Definition of Union)} \\
&\iff (x \in A \land y \in B) \lor (x \in A \land y \in C) && \text{(Logical Distributivity)} \\
&\iff (x,y) \in (A \times B) \lor (x,y) \in (A \times C) \\
&\qquad \text{(Definition of Cartesian Product)} \\
&\iff (x,y) \in (A \times B) \cup (A \times C) && \text{(Definition of Union)}
\end{aligned}$$

Since every arbitrary pair belongs to the left set if and only if it belongs to the right set, the sets are equal. ■

1.7 The Relative Complement

Theorem 1.5 Show that $A \setminus (A \setminus B) = A \cap B$.

Intuition: If you take away from A everything in A that is not in B, you are left exactly with the part of A that IS in B.

Proof. Let x be an arbitrary element. We establish logical equivalence.

$$\begin{aligned}
x \in A \setminus (A \setminus B) &\iff x \in A \land x \notin (A \setminus B) && \text{(Definition of Difference)} \\
&\iff x \in A \land \neg(x \in A \land x \notin B) && \text{(Definition of Difference)} \\
&\iff x \in A \land (x \notin A \lor x \in B) && \text{(De Morgan's Law of Logic)} \\
&\iff (x \in A \land x \notin A) \lor (x \in A \land x \in B) && \text{(Logical Distributivity)} \\
&\iff \text{False} \lor (x \in A \land x \in B) && \text{(Law of Non-Contradiction)} \\
&\iff x \in A \land x \in B && \text{(Identity Law for Disjunction)}
\end{aligned}$$

$$\iff x \in A \cap B \quad \text{(Definition of Intersection)}$$

Since both sets contain exactly the same elements, $A \setminus (A \setminus B) = A \cap B$. ∎

1.8 De Morgan's Law for Set Difference over Intersection

Theorem 1.6 Show that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Intuition: Removing the intersection of B and C from A means you remove elements only if they are in BOTH. Thus, if an element in A misses B or misses C, it stays.

Proof. Let x be an arbitrary element

$$\begin{aligned}
x \in A \setminus (B \cap C) &\iff x \in A \land x \notin (B \cap C) && \text{(Definition of Difference)} \\
&\iff x \in A \land \neg(x \in B \land x \in C) && \text{(Definition of Intersection)} \\
&\iff x \in A \land (x \notin B \lor x \notin C) && \text{(De Morgan's Law of Logic)} \\
&\iff (x \in A \land x \notin B) \lor (x \in A \land x \notin C) && \text{(Logical Distributivity)} \\
&\iff (x \in A \setminus B) \lor (x \in A \setminus C) && \text{(Definition of Difference)} \\
&\iff x \in (A \setminus B) \cup (A \setminus C) && \text{(Definition of Union)}
\end{aligned}$$

Thus, the sets are equal. ∎

1.9 Right Distributivity of Difference over Union

Theorem 1.7 Show that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

Proof. Let x be an arbitrary element

$$\begin{aligned}
x \in (A \cup B) \setminus C &\iff x \in (A \cup B) \land x \notin C && \text{(Definition of Difference)} \\
&\iff (x \in A \lor x \in B) \land x \notin C && \text{(Definition of Union)} \\
&\iff (x \in A \land x \notin C) \lor (x \in B \land x \notin C) && \text{(Logical Distributivity)} \\
&\iff (x \in A \setminus C) \lor (x \in B \setminus C) && \text{(Definition of Difference)} \\
&\iff x \in (A \setminus C) \cup (B \setminus C) && \text{(Definition of Union)}
\end{aligned}$$

This proves the right-distributivity of set subtraction. ∎

1.10 Intersection with a Difference

Theorem 1.8 Show that $A \cap (B \setminus C) = (A \cap B) \setminus C = (A \cap B) \setminus (A \cap C)$.

Proof. Proof of First Equality: Let x be an arbitrary element

$$\begin{aligned}
x \in A \cap (B \setminus C) &\iff x \in A \land x \in (B \setminus C) \\
&\iff x \in A \land (x \in B \land x \notin C) \\
&\iff (x \in A \land x \in B) \land x \notin C \quad \text{(Associativity of Conjunction)} \\
&\iff x \in (A \cap B) \land x \notin C \\
&\iff x \in (A \cap B) \setminus C
\end{aligned}$$

Proof of Second Equality: Continuing from the middle step:

$$\begin{aligned}
(x \in A \land x \in B) \land x \notin C &\iff (x \in A \land x \in B) \land (\text{True} \land x \notin C) \\
&\iff (x \in A \land x \in B) \land (x \notin A \lor x \notin C) \\
&\text{(Since } x \in A \text{ is true, } x \notin A \text{ is false)} \\
&\iff (x \in A \land x \in B) \land \neg(x \in A \land x \in C) \\
&\iff x \in (A \cap B) \land x \notin (A \cap C) \\
&\iff x \in (A \cap B) \setminus (A \cap C)
\end{aligned}$$

Thus, all three sets are equal. ∎

1.11 The Absorption Laws

The Absorption Laws are crucial for simplifying complex expressions where one set "swallows" another. There are two forms.

Theorem 1.9 Show that (Union absorbs Intersection): $A \cup (A \cap B) = A$.

Proof. Let x be an arbitrary element.

$$\begin{aligned}
&x \in A \cup (A \cap B) \\
&\iff x \in A \lor (x \in A \land x \in B) && \text{(Definitions of Union and Intersection)} \\
&\iff (x \in A \land \text{True}) \lor (x \in A \land x \in B) && \text{(Logical Identity)} \\
&\iff x \in A \land (\text{True} \lor x \in B) && \text{(Logical Distributivity extracted)} \\
&\iff x \in A \land \text{True} && \text{(Domination Law: True } \lor P \equiv \text{True)} \\
&\iff x \in A && \text{(Logical Identity)}
\end{aligned}$$

Thus, $A \cup (A \cap B) = A$. ∎

Theorem 1.10 (Intersection absorbs Union): Show that $A \cap (A \cup B) = A$.

Proof. Let x be an arbitrary element.

$$\begin{aligned}
&x \in A \cap (A \cup B) \\
&\iff x \in A \wedge (x \in A \vee x \in B) \\
&\iff (x \in A \vee \text{False}) \wedge (x \in A \vee x \in B) \\
&\iff x \in A \vee (\text{False} \wedge x \in B) && \text{(Logical Distributivity extracted)} \\
&\iff x \in A \vee \text{False} && \text{(Domination Law: False } \wedge P \equiv \text{False)} \\
&\iff x \in A
\end{aligned}$$

Thus, $A \cap (A \cup B) = A$. ∎

1.12 Difference of a Union (Reverse Distributivity)

Theorem 1.11 Show that $A \setminus (A \cup B) = \emptyset$.

Intuition: If you start with A and remove everything in A AND everything in B, there is nothing left of A.

Proof. Let x be an arbitrary element. We check the membership condition for the left side

$$\begin{aligned}
x \in A \setminus (A \cup B) &\iff x \in A \wedge x \notin (A \cup B) \\
&\iff x \in A \wedge \neg(x \in A \vee x \in B) \\
&\iff x \in A \wedge (x \notin A \wedge x \notin B) && \text{(De Morgan's Law)} \\
&\iff (x \in A \wedge x \notin A) \wedge x \notin B && \text{(Associativity)} \\
&\iff \text{False} \wedge x \notin B && \text{(Contradiction)} \\
&\iff \text{False}
\end{aligned}$$

Because the statement $x \in A \setminus (A \cup B)$ is logically false for any x, the set contains no elements. Therefore, it is the empty set $\emptyset$. ∎

2 Algebraic Set Theory: Direct Proofs (Without Elements)

Direct proofs in set theory avoid element-chasing (choosing an arbitrary x) by relying instead on the established algebraic properties of set operations, often called the Boolean Algebra of sets.

To perform these proofs, we define a universal set U, let A^c denote the absolute complement of A, and rewrite set difference purely in terms of intersection and complementation:

$$A \setminus B = A \cap B^c$$

The fundamental laws utilized below include:

1. **De Morgan's Laws:** $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

2. **Distributive Laws:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
3. **Complement Laws:** $A \cap A^c = \emptyset$, $A \cup A^c = U$, and $(A^c)^c = A$
4. **Identity & Domination:** $A \cap U = A$, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$, and $A \cup U = U$

2.1 The Relative Complement of a Relative Complement

Theorem 2.1 Show that $A \setminus (A \setminus B) = A \cap B$.

Proof.

$$\begin{aligned}
A \setminus (A \setminus B) &= A \setminus (A \cap B^c) && \text{(Definition of Set Difference)} \\
&= A \cap (A \cap B^c)^c && \text{(Definition of Set Difference)} \\
&= A \cap (A^c \cup (B^c)^c) && \text{(De Morgan's Law)} \\
&= A \cap (A^c \cup B) && \text{(Double Complement Law)} \\
&= (A \cap A^c) \cup (A \cap B) && \text{(Distributive Law)} \\
&= \emptyset \cup (A \cap B) && \text{(Complement Law)} \\
&= A \cap B && \text{(Identity Law)}
\end{aligned}$$

■

2.2 De Morgan's Law for Set Difference over Intersection

Theorem 2.2 Show that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof.

$$\begin{aligned}
A \setminus (B \cap C) &= A \cap (B \cap C)^c && \text{(Definition of Set Difference)} \\
&= A \cap (B^c \cup C^c) && \text{(De Morgan's Law)} \\
&= (A \cap B^c) \cup (A \cap C^c) && \text{(Distributive Law)} \\
&= (A \setminus B) \cup (A \setminus C) && \text{(Definition of Set Difference)}
\end{aligned}$$

■

2.3 Right Distributivity of Difference over Union

Theorem 2.3 Show that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.